QUANTUM LADDER OPERATORS & LIE ALGEBRAS

CREATION, ANNIHILATION, & $\sigma_{\pm}$

Commutation Relations and State Vector Space Mapping

An axiomatic guide detailing bosonic fields, two-level ladder operators, and their underlying algebraic structures.

1. INFINITE-DIMENSIONAL BOSONIC OPERATORS $(A, A^{\dagger})$

In the Quantum Harmonic Oscillator and Quantized Electromagnetic Fields, states are built in an infinite-dimensional Fock Space using the annihilation operator a and creation operator $a^{\dagger}$.

State Transformations **Number Operator:** Acting on an n-photon basis state $|n\rangle$ scales the vector by the eigenvalue square root:

$$a\,|n\rangle = \sqrt{n}\,|n-1\rangle \qquad a^{\dagger}\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \qquad a\,|0\rangle = 0 \text{ (Vacuum)}$$

The Number Operator: $N = a^{\dagger}a$ yields the discrete excitation count: $N\,|n\rangle = a^{\dagger}a\,|n\rangle = n\,|n\rangle$

Bosonic Commutation Relations: Bosons satisfy the Canonical Commutation Relations (CCR), which form the foundation of quantum field statistics and field uncertainty:

$$[a, a^{\dagger}] = aa^{\dagger} - a^{\dagger}a = I \qquad [N, a] = -a \qquad [N, a^{\dagger}] = a^{\dagger} \qquad [a, a] = [a^{\dagger}, a^{\dagger}] = 0$$

The non-zero identity result $[a, a^{\dagger}] = 1$ is the algebraic origin of vacuum zero-point energy: $H = \hbar\omega(N + \frac{1}{2})$.

2. FINITE-DIMENSIONAL QUBIT OPERATORS (σ_{+}, σ_{-})

For a discrete two-level system (qubit) spanned by basis states $\{|g\rangle, |e\rangle\}$ or $\{|0\rangle, |1\rangle\}$, the ladder operations are executed by the atomic raising (σ_{+}) and lowering (σ_{-}) operators.

Matrix Formulations in the Computational Basis: Written as combinations of Pauli X and Y operators:

$$\sigma_{+} = \frac{X + iY}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |e\rangle\langle g| \qquad \sigma_{-} = \frac{X - iY}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |g\rangle\langle e|$$

- **Raising Transformation:** $\sigma_{+}\,|g\rangle = |e\rangle$ and $\sigma_{+}\,|e\rangle = 0$ *(Cannot excite a state past its maximum manifold)*
- **Lowering Transformation:** $\sigma_{-}\,|e\rangle = |g\rangle$ and $\sigma_{-}\,|g\rangle = 0$ *(Cannot decay lower than the ground state)*

3. THE PAULI OPERATOR GOTCHA: COMMUTATION VS. ANTI-COMMUTATION

While bosonic operators commute ($[a, a^{\dagger}] = 1$), single-qubit ladder operators behave like fermions locally, obeying **anti-commutation** relationships due to the exclusion of multiple excitations on a single node.

SU(2) LIE ALGEBRA AND ALGEBRAIC BOUNDARIES

The commutation relation for $\sigma_{\pm}$ yields the diagonal Pauli Z matrix, mapping state inversion:

$$[\sigma_{+}, \sigma_{-}] = \sigma_{+}\sigma_{-} - \sigma_{-}\sigma_{+} = |e\rangle\langle e| - |g\rangle\langle g| = Z$$

Local Fermionic Anti-Commutation Constraint: $\{\sigma_{+}, \sigma_{-}\} = \sigma_{+}\sigma_{-} + \sigma_{-}\sigma_{+} = I$

The Nilpotency Gotcha: Because a single qubit cannot absorb two excitations, the operators are nilpotent: $\sigma_{+}^{\mathbf{2}} = \sigma_{-}^{\mathbf{2}} = \mathbf{0}$.

4. SUMMARY OF COMMUTATOR INTERACTIONS

The interactions between excitation tracking and ladder operators define the dynamic equations of movement within the Jaynes-Cummings model circuit space:

Bosonic Commutator:	$[a, a^{\dagger}] = 1$	Generates the infinite ladder of photon states.
Atomic Commutator:	$[\sigma_{+}, \sigma_{-}] = Z$	Controls the closed two-level spin inversion loops.
Cross-Commutator:	$[Z, \sigma_{\pm}] = \pm 2\sigma_{\pm}$	Drives the active energy shifts in driven qubit dynamics.