QUANTUM INFORMATION THEORY & RESOURCE MEASURES

ENTANGLEMENT QUANTIFICATION

Von Neumann Entropy, Schmidt Decompositions, and Subsystem Mixedness

An axiomatic guide detailing the algebraic measures used to calculate, bound, and verify non-local multi-partite resources.

1. OPERATIONAL DEFINITION: QUANTUM INFORMATION ENTROPY

Von Neumann entropy scales classical Shannon information entropy into the domain of non-commutative operators. It acts as the primary tool for measuring the structural disorder, uncertainty, or mixedness contained within a quantum density matrix ρ.

The Von Neumann Entropy Formulation: Calculated using the functional trace profile of the global operator or by isolating its discrete eigenvalues λ_i:

$$S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho) = -\sum_i \lambda_i \log_2 \lambda_i$$

Axiomatic Properties:

✦ **Pure State Constraint:** If $\rho = |\psi\rangle\langle\psi|$, it possesses exactly one non-zero eigenvalue ($\lambda_1 = 1$). This forces $S(\rho) = 0$.

✦ **Maximally Mixed Bound:** For a d-dimensional space, $S(\rho)$ maximizes at $\log_2 d$ when $\rho = \frac{1}{d} I$.

2. BIPARTITE RESOURCE METRICS: ENTROPY OF ENTANGLEMENT

For a global composite system AB preserved in a pure state $|\psi\rangle_{AB}$, any non-local correlations prevent the total vector from being factored into isolated product states ($|\psi\rangle_A \otimes |\psi\rangle_B$). This non-separability forces individual subsystems to behave like mixed states.

The Measurement Mechanic: To evaluate the precise **Entropy of Entanglement** (E), take the partial trace to isolate the reduced density operator of a single subsystem, then evaluate its Von Neumann profile:

$$\rho_A = \mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|) \qquad \Longrightarrow \qquad E(|\psi\rangle_{AB}) = S(\rho_A) = S(\rho_B)$$

The identity $S(\rho_A) = S(\rho_B)$ holds strictly true for any global pure state, proving that subsystem uncertainties are perfectly balanced.

3. THE ALGEBRAIC CORE: SCHMIDT DECOMPOSITION

Calculating the reduced eigenvalues λ_i is simplified using the **Schmidt Decomposition theorem**. Any bipartite pure state can be written in a diagonal matrix layout across specific local orthonormal bases $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$:

THE SCHMIDT RESOURCE FORMULATION

The vector maps directly to a single summation loop bounded by the Schmidt rank $k \leq \min(d_A, d_B)$:

$$|\psi\rangle_{AB} = \sum_{i=1}^{k} \sqrt{\lambda_i}\, |i\rangle_A |i\rangle_B \qquad \text{where} \qquad \sum_{i=1}^{k} \lambda_i = 1, \quad \lambda_i > 0$$

The Direct Measure Connection: The squared coefficients λ_i match the eigenvalues of the reduced density operators ρ_A and ρ_B. If $k = 1$, the state is cleanly separable ($E = 0$). If $k > 1$, the system is natively entangled.

4. MEASURING STRUCTURAL CONFIGURATIONS

Applying these entropy and decomposition pipelines reveals the resource depth of leading quantum computing configurations:

✦ **Maximally Entangled Bell Pair:** Taking $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ yields equal Schmidt coefficients $\lambda_1 = \lambda_2 = \frac{1}{2}$. Tracing a qubit leaves $\rho_A = \frac{1}{2} I$. The resulting entropy of entanglement scales to exactly **1 Bit** ($E = -\frac{1}{2}\log_2 \frac{1}{2} - \frac{1}{2}\log_2 \frac{1}{2} = 1$), representing one fundamental **Ebit** of quantum resource.

✦ **The 3-Qubit GHZ State:** For $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, bipartitioning the system along the boundary $A|BC$ drops the localized operator to $\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. The entropy measures exactly $S(\rho_A) = 1$. This proves that

despite scaling out to larger registers, the maximum bipartite entanglement across any single qubit boundary remains fixed at 1 Ebit.