LINEAR OPERATORS & MATRIX MECHANICS

# ESSENTIAL MATRIX OPERATIONS

## Properties, Transformations, and Structural Metrics in Quantum Information

An axiomatic guide detailing the algebraic rules, matrix decompositions, and properties that dictate quantum state evolution.

## 1. THE FOUNDATIONAL MAPPING: CONJUGATE TRANSPOSE (ADJOINT)

Every state manipulation and physical observable is governed by the matrix adjoint (conjugate transpose), denoted by †. For an arbitrary complex matrix $A$, it maps entry-wise as the transpose of the complex conjugate elements:

**Algebraic Adjoint Properties:** Given matrices $A$ and $B$, and a complex scalar $\alpha \in \mathbb{C}$:

$$A^\dagger = (A^*)^T \qquad \Longrightarrow \qquad (A_{ij})^\dagger = A^*_{ji}$$

**Reversal Properties:** $(AB)^\dagger = B^\dagger A^\dagger$ and $(\alpha A)^\dagger = \alpha^* A^\dagger$

## 2. ESSENTIAL MATRIX CLASSES PHYSICS MANIFESTATIONS

The algebraic definitions of matrix classes directly impose the fundamental conservation laws of quantum mechanics:

- **Hermitian Matrices ($H^\dagger = H$):** Map physical observables. Spectral theory guarantees that their eigenvalues are purely real ($\lambda_i \in \mathbb{R}$), matching actual physical readouts. Their eigenvectors form an orthonormal basis.
- **Unitary Matrices ($U^\dagger U = UU^\dagger = I$):** Map closed-system evolution (gates). They preserve the vector norm ($\|U|\psi\rangle\| = \||\psi\rangle\|$), ensuring total probability remains identically 1. Their eigenvalues lie strictly on the complex unit circle ($e^{i\theta}$).
- **Positive Semi-Definite Matrices ($M \geq 0$):** Satisfy $\langle\psi| M |\psi\rangle \geq 0$ for all vectors. This property is mandatory for **Density Matrices** ($\rho$), guaranteeing that state measurement probabilities can never be negative.

## 3. SUBSYSTEM ISOLATION: THE PARTIAL TRACE ($\mathrm{TR}_B$)

When evaluating composite entangled systems $\mathcal{H}_A \otimes \mathcal{H}_B$, local properties cannot be read using the global trace. Isolating the local density matrix of system $A$ requires calculating the **Partial Trace**, which averages out (discards) the degrees of freedom of system $B$.

**The Linear Operator Mapping:** Let $|i_A\rangle, |j_A\rangle$ be basis vectors of space $A$, and $|k_B\rangle, |l_B\rangle$ be basis vectors of space $B$. The partial trace of a composite basis operator scales as:

$$\mathrm{Tr}_B\left(|i_A\rangle\langle j_A| \otimes |k_B\rangle\langle l_B|\right) = |i_A\rangle\langle j_A| \ \mathrm{Tr}(|k_B\rangle\langle l_B|) = |i_A\rangle\langle j_A|\langle l_B|k_B\rangle = |i_A\rangle\langle j_A|\,\delta_{kl}$$

*Applying this operation to an entangled Bell state collapses the cross-diagonal phase terms, exposing local mixedness.*

## 4. MATRIX FUNCTIONS & THE SPECTRAL THEOREM

Evaluating complex operations on quantum networks (such as time evolution $e^{-iHt/\hbar}$ or computing Von Neumann entropy $-\rho\log_2\rho$) requires calculating functions of large matrices. This is enabled by the **Spectral Theorem** for normal operators ($[A, A^\dagger] = 0$).

**FUNCTIONAL CALCULUS ON MATRIX CONSTRUCTS**

Any normal operator can be diagonalized using its eigenvalues $\lambda_i$ and its orthonormal projection operators $P_i = |i\rangle\langle i|$:

$$A = \sum_i \lambda_i |i\rangle\langle i| \qquad \Longrightarrow \qquad \textbf{Matrix Function Law:} \quad f(A) = \sum_i f(\lambda_i)\,|i\rangle\langle i|$$

**The Computational Execution:** Instead of computing infinite matrix power series expansions, the transformation maps the function $f$ directly onto the discrete scalar eigenvalues, preserving the structural spatial projections.

## 5. STRUCTURAL METRICS: COMMUTATORS FUNCTIONS

The interaction properties of combined matrix blocks dictate precision and information containment boundaries:

1. **The Trace (Tr):** Sum of the diagonal entries ($\sum_i A_{ii}$). It is invariant under basis changes and cyclic permutations ($\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA)$).

2. **The Commutator ($[A, B] = AB - BA$):** Measures non-commutativity. If $[A, B] \neq 0$, the matrices cannot be simultaneously diagonalized, generating physical uncertainty.
3. **The Anti-Commutator ($\{A, B\} = AB + BA$):** Governs fermionic statistics and updates mapping configurations in error correction.