DIFFERENTIAL OPERATORS, MANY-BODY FIELD THEORY, & PROPAGATORS

GREEN'S FUNCTIONS IN QUANTUM

Impulse Responses, Time-Ordered Propagators, and Perturbation Theory

An axiomatic mathematical guide detailing how Green's functions solve interacting quantum differential equations.

1. THE CLASSICAL TO QUANTUM PIVOT: THE IMPULSE RESPONSE

Mathematically, a Green's function $G(x, x')$ is the impulse response of a linear differential operator L. It defines the field generated at coordinate x by a localized delta-function point source acting at x':

The Defining Operator Differential Equation:
Given a linear differential equation $L\psi(x) = f(x)$, the auxiliary Green's function equation matches:

$$L\,G(x, x') = \delta(x - x') \qquad \Longrightarrow \qquad \psi(x) = \int G(x, x')f(x')\,dx'$$

The Quantum Operator Inversion Limit:
In quantum mechanics, where L maps to an operator containing the Hamiltonian $(E - H)$, the Green's function acts directly as the matrix inverse of the differential equation:

$$G(E) = (E - H)^{-1}$$

2. SINGLE-PARTICLE QUANTUM PROPAGATORS

In time-dependent quantum mechanics, Green's functions map how a state vector propagates dynamically through space and time.

- **Retarded Green's Function (G^R):**
  Governs forward-in-time causal propagation:

$$G^R(x, t; x', 0) = -i\Theta(t)\,\langle 0|\,\{\psi(x, t), \psi^\dagger(x', 0)\}\,|0\rangle$$

- **Advanced Green's Function (G^A):**
  Governs backward-in-time propagation, dropping to zero for $t > 0$.

3. MANY-BODY FIELD THEORY: TIME-ORDERED GREEN'S FUNCTIONS

THE TIME-ORDERED PROPAGATOR PATTERN

$$G(x, t; x', t') = -i\,\langle\Omega|\,\mathcal{T}[\psi(x, t)\psi^\dagger(x', t')]\,|\Omega\rangle$$

$$\mathcal{T}[\psi(t)\psi^\dagger(t')] = \Theta(t - t')\psi(t)\psi^\dagger(t') \pm \Theta(t' - t)\psi^\dagger(t')\psi(t)$$

($-$ for Fermions, $+$ for Bosons)

Physical Insight:
For $t > t'$, G maps the propagation of an injected electron. For $t < t'$, it maps the propagation of a hole excitation.

4. SOLVING INTERACTIONS: THE DYSON EQUATION

Green's functions provide the foundational framework for perturbation theory.
The Matrix Dyson Transformation:
Introducing the self-energy operator Σ:

$$G = G_0 + G_0\Sigma G_0 + G_0\Sigma G_0\Sigma G_0 + \dots$$

which compactly becomes

$$G = G_0 + G_0\Sigma G$$

This recursive algebraic structure allows highly interacting systems to be represented through Feynman-diagram expansions.

5. EXTRACTING PHYSICAL METRICS: THE LEHMANN REPRESENTATION

By inserting a complete set of eigenstates:

$$\sum_n |n\rangle \langle n| = I$$

the Green's function becomes

$$G(k,\omega) = \sum_n \frac{|\langle n| \psi^\dagger(0) |\Omega\rangle|^2}{\omega - (E_n - E_\Omega) + i\eta} + \sum_m \frac{|\langle m| \psi(0) |\Omega\rangle|^2}{\omega - (E_\Omega - E_m) - i\eta}$$

Spectral Function:

$$A(k,\omega) = -\frac{1}{\pi} \operatorname{Im} G^R(k,\omega)$$

The spectral function maps the density of states and quasiparticle excitation spectrum.