THE LANGUAGE OF QUANTUM MECHANICS

LINEAR ALGEBRA FOR QUANTUM

Vector Spaces, Inner Products, Operators, and Spectral Formulations

An axiomatic mathematical guide mapping core geometric constructs to the physics of quantum information.

1. THE CANVAS: HILBERT SPACES & DIRAC NOTATION

Quantum states exist as vectors within a complex inner product space called a **Hilbert Space** ($\mathcal{H}$). Dirac notation abstracts these vectors into highly scannable notation elements.

Kets, Bras, and Inner Products:

✦ **Ket $|\psi\rangle$:** A column vector representing the quantum state: $|\psi\rangle = \sum_i c_i |i\rangle \longleftrightarrow \begin{pmatrix} c_1 & c_2 & \dots & c_d \end{pmatrix}^T$.

✦ **Bra $\langle\psi|$:** The conjugate transpose (adjoint) row vector: $\langle\psi| = |\psi\rangle^\dagger = \begin{pmatrix} c_1^* & c_2^* & \dots & c_d^* \end{pmatrix}$.

✦ **Inner Product $\langle\phi|\psi\rangle$:** Maps two vectors to a complex scalar ($\mathbb{C}$). It satisfies conjugate symmetry $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$ and determines the state overlap.

The Norm: The physical requirement of probability conservation restricts state space trajectories to unit vectors residing on the hypersphere boundary: $\||\psi\rangle\| = \sqrt{\langle\psi|\psi\rangle} = 1$.

2. LINEAR OPERATORS & OUTER PRODUCTS

A linear operator A maps vectors onto alternative configurations within the same space: $A|\psi\rangle = |\psi'\rangle$.

- **Outer Product $|\phi\rangle\langle\psi|$:** Multiplies a column vector by a row vector to generate a valid $d \times d$ linear matrix.
- **Resolution of the Identity:** Given an orthonormal basis $\{|i\rangle\}$, the sum of their parallel projection matrices fills the identity space completely: $\sum_i |i\rangle\langle i| = I$. This allows any operator to be decomposed as $A = IAI = \sum_{i,j} A_{ij} |i\rangle\langle j|$, where $A_{ij} = \langle i| A |j\rangle$.

3. THE CORE ENGINE: ADJOINTS AND THE SPECTRAL THEOREM

The structural behavior of an operator is governed by its matrix adjoint (conjugate transpose $A^\dagger$). Two critical matrix classes partition quantum kinematics:

Hermitian Operators ($A^\dagger = A$): Map physical observables. The **Spectral Theorem** guarantees that any Hermitian matrix can be written in a perfectly diagonalized configuration using its real eigenvalues λ_i and its orthonormal projections:

$$A = \sum_i \lambda_i |i\rangle\langle i| \qquad \text{where } \lambda_i \in \mathbb{R} \quad \text{and} \quad \langle i|j\rangle = \delta_{ij}$$

Unitary Operators ($U^\dagger U = UU^\dagger = I$): Map closed-system evolutions and logic gates. They preserve the global metric length $\langle\psi| U^\dagger U |\psi\rangle = \langle\psi|\psi\rangle = 1$, ensuring probability maps accurately during transitions.

4. MULTI-PARTICLE SCALING: THE KRONECKER PRODUCT ($\otimes$)

When multiple isolated physical systems combine, their individual Hilbert spaces do not add; they multiply exponentially via the **Kronecker (Tensor) Product**: $\mathcal{H}_{\text{total}} = \mathcal{H}_1 \otimes \mathcal{H}_2 \cdots \otimes \mathcal{H}_N$.

MATRIX DIMENSION ACCELERATION

For individual matrix elements A $(m \times n)$ and B $(p \times q)$, the tensor layout generates an expanded $(mp) \times (nq)$ structural grid:

$$A \otimes B = \begin{pmatrix} A_{11}B & \dots & A_{1n}B \\ \vdots & \ddots & \vdots \\ A_{m1}B & \dots & A_{mn}B \end{pmatrix}$$

The Entanglement Separation Limit: A combined global state vector $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is classified as unentangled (separable) if it can be broken down as $|\psi\rangle_A \otimes |\phi\rangle_B$. If no such decomposition exists, the state is natively entangled.

5. FUNDAMENTAL MATRIX METRICS: THE TRACE AND THE COMMUTATOR

Two primary algebraic evaluations map physical operations to concrete numeric boundaries:

1. **The Trace (Tr):** The sum of the diagonal matrix entries: $\text{Tr}(A) = \sum_i A_{ii} = \sum_i \langle i| A |i\rangle$. It is invariant under cyclic shifts ($\text{Tr}(ABC) = \text{Tr}(CAB)$) and forms the basis for expectation values: $\langle A\rangle = \text{Tr}(\rho A)$.

2. **The Commutator ($[A, B] = AB - BA$):** Quantifies matrix non-commutativity. If $[A, B] \neq 0$, the corresponding matrices do not share a common basis, preventing concurrent deterministic measurements.