DEGREES OF FREEDOM & MULTI-DIMENSIONAL PHOTONIC ENGINEERING

PHOTON ENTANGLEMENT TYPES

Polarization, Time-Energy, Orbital Angular Momentum, and Hyperentanglement

An axiomatic mathematical guide mapping the explicit physical degrees of freedom used to distribute non-local quantum resources.

1. THE FOUNDATION: ENGINEERING MULTI-PARTICLE STATES VIA SPDC

Photonic entanglement is the core engine of quantum information networks. The primary source for these states is **Spontaneous Parametric Down-Conversion (SPDC)**, where a higher-energy pump photon splits inside a non-linear crystal into a pair of entangled photons (Signal and Idler). The global state must preserve energy ($\omega_p = \omega_s + \omega_i$) and momentum ($\vec{k}_p = \vec{k}_s + \vec{k}_i$), allowing non-local correlations to be tailored across distinct physical degrees of freedom.

2. POLARIZATION ENTANGLEMENT (DISCRETE 2D SPACES)

The most widely deployed type of photonic entanglement utilizes the internal transverse polarization vector of light. It naturally operates in a discrete two-dimensional Hilbert space ($\mathcal{H}_2$), mapping perfect correlations across orthogonal directions.

The Polarization Bell Basis Formulations: Let $|H\rangle$ represent horizontal and $|V\rangle$ represent vertical single-photon polarization states. The maximally entangled states map as:

$$|\Phi^{\pm}\rangle = \frac{|HH\rangle \pm |VV\rangle}{\sqrt{2}} \qquad \text{and} \qquad |\Psi^{\pm}\rangle = \frac{|HV\rangle \pm |VH\rangle}{\sqrt{2}}$$

Operational Features: Easily manipulated using waveplates and polarized beam splitters. This architecture forms the default standard for the BB84 and E91 QKD protocols, as well as early quantum teleportation demonstrations.

3. TIME-ENERGY & TIME-BIN ENTANGLEMENT (CONTINUOUS TO DISCRETE MAPPINGS)

Derived directly from the continuous creation-time correlations of SPDC photon pairs, **Time-Energy Entanglement** implies that neither photon has a defined birth time, but they are born exactly simultaneously.

The Continuous Mode State: $|\Psi\rangle = \int d\omega\, c(\omega)\, |\omega_0 + \omega\rangle_s\, |\omega_0 - \omega\rangle_i$
To convert this continuous variable into a robust discrete setup for fiber routing, light is passed through unbalanced Mach-Zehnder interferometers, carving out distinct **Time-Bin Entangled** slots:

$$|\Psi\rangle_{\text{time-bin}} = \frac{|\text{Early}\rangle_s\, |\text{Early}\rangle_i + e^{i\phi}\, |\text{Late}\rangle_s\, |\text{Late}\rangle_i}{\sqrt{2}}$$

Because single-photon polarization vectors experience rapid depolarization drift inside long-distance telecom fibers due to random stress variations, time-bin encodings serve as the baseline architecture for multi-kilometer quantum networks.

4. ORBITAL ANGULAR MOMENTUM (OAM) ENTANGLEMENT (INFINITE DISCRETE DIMENSIONS)

While spin angular momentum dictates circular polarization states ($\pm\hbar$), photons passing through structured phase profiles (like spiral phase plates) acquire an azimuthal phase factor $e^{il\phi}$. This factor grants them **Orbital Angular Momentum (OAM)** equivalent to $l\hbar$. Because the topological charge l is an integer ($l \in \mathbb{Z}$), OAM opens an **infinite-dimensional discrete Hilbert space**.

HIGH-DIMENSIONAL QUANTUM CORRELATIONS

The down-conversion phase-matching conditions enforce absolute conservation of OAM across the emerging particle states:

$$|\Psi\rangle_{\text{OAM}} = \sum_{l=-\infty}^{\infty} c_l \, |l\rangle_s \, |-l\rangle_i$$

The Scalability Advantage: Instead of compressing quantum keys into standard binary qubits, OAM generates high-dimensional **qudits**. This expands the information density per photon from 1 bit to $\log_2 d$ bits, dramatically augmenting the noise tolerance and security thresholds of long-range free-space communication channels.

5. HYPERENTANGLEMENT & PATH-LENGTH MULTI-ENCODING

When single-photon pairs are generated such that they are simultaneously entangled across **multiple completely independent** degrees of freedom, the system is classified as **Hyperentangled**.

The Product Tensor Space Mapping: If a pair is synchronously correlated in polarization (P), time-bin (T), and spatial path (K), the global wave function maps as the tensor product of those individual resource structures:

$$|\Psi\rangle_{\text{global}} = |\Psi\rangle_P \otimes |\Psi\rangle_T \otimes |\Psi\rangle_K \in \mathcal{H}_P \otimes \mathcal{H}_T \otimes \mathcal{H}_K$$

The Quantum Edge: This multidimensional coupling allows for complete, deterministic **Bell state measurements** using basic linear optics, completely bypassing standard structural measurement boundaries. It serves as a primary performance booster for superdense coding and advanced multi-particle quantum repeaters.