QUANTUM INFORMATION LIMITS

QUANTUM IMPOSSIBILITY THEOREMS

Fundamental Constraints on Quantum Information Processing

An axiomatic flyer mapping the algebraic restrictions that protect and define quantum structures.

1. NO-BROADCASTING THEOREM (THE GENERALISED NO-CLONING)

While the No-Cloning theorem forbids duplication of pure states, the **No-Broadcasting Theorem** extends this restriction to arbitrary mixed states ρ. It proves that a quantum channel cannot take a single mixed state and output two separate states that match the original marginals.

The Mathematical Bound: Given an unknown density matrix ρ, it is impossible to find a universal valid quantum operation $\mathcal{E}$ capable of broadcasting it onto a target channel space:

$$\mathcal{E}(\rho \otimes \theta) = \widetilde{\rho} \quad \text{such that} \quad \mathrm{Tr}_1(\widetilde{\rho}) = \rho \quad \text{and} \quad \mathrm{Tr}_2(\widetilde{\rho}) = \rho$$

The Constraint: Broadcasting is achievable if and only if the set of input states belongs to a mutually commuting family ($[\rho_i, \rho_j] = 0$).

2. NO-DELETION THEOREM (THE TIME-REVERSE COUNTERPART)

Formulated by Pati and Braunstein in 2000, the **No-Deletion Theorem** serves as the precise time-reverse mirror of No-Cloning. It states that given two identical copies of an unknown quantum state, it is physically impossible to erase one copy using a linear quantum process.

The Algebraic Mechanism: Assume a unitary process U attempts to overwrite one copy of $|\psi\rangle$ given a blank state $|e\rangle$:

$$U\,|\psi\rangle\,|\psi\rangle\,|A\rangle = |\psi\rangle\,|e\rangle\,|A_\psi\rangle$$

Linearity forces the final state of the auxiliary apparatus $|A_\psi\rangle$ to retain a perfect, permanent record of the state geometry. Thus, the information is never truly destroyed or deleted; it is merely shuffled into environmental degrees of freedom.

3. NO-HIDING THEOREM (CONSERVATION OF QUANTUM INFORMATION)

The **No-Hiding Theorem** enforces that quantum information can never be truly hidden in the correlations between a system and its environment. If information disappears from a local subsystem via decoherence, it must completely leak into the environment.

THE CONSERVATION PILE

Let a state $|\psi\rangle$ undergo a process where its sub-space representation maps into a completely invariant state $|0\rangle$, hiding it from local observation:

$$\mathcal{U}\,|\psi\rangle_S\,|0\rangle_E = |0\rangle_S\,|\Phi(\psi)\rangle_E$$

The Consequence: The transformation requires the map to the environment state $|\Phi(\psi)\rangle_E$ to be perfectly isometric. The original information remains completely intact, structured inside the environment, protecting information from absolute loss.

4. THE NO-TELEPORTATION LIMIT (WITHOUT CLASSICAL SIGNALLING)

This limit theorem establishes that quantum states cannot be disassembled and reconstructed remotely without explicit coordinate adjustments over an accompanying classical channel.

- **The Bound:** An entangled pair allows Alice to project a combined state, but Bob's remote outcome remains a uniformly mixed random rotation: $\rho_B = \frac{1}{2}I$.
- **Resolution:** Bob requires exactly 2 classical bits from Alice to choose the proper corrective Pauli matrix (X, Y, Z, or I) to isolate the clean state vector.