QUANTUM COMPUTING AND CIRCUIT ARCHITECTURES

QUANTUM LOGIC GATES

Mathematical Transformations in Multi-Qubit Circuits

An axiomatic guide to unitary operations, matrix representations, and universal gate sets.

1. MATHEMATICAL FOUNDATIONS OF QUANTUM GATES

Quantum gates are mathematical operators that transform a quantum state. Because physical systems must conserve total probability (Postulate 4), every closed-system quantum gate is modeled as a **Unitary Matrix** U.

Unitary Condition: $U^\dagger U = UU^\dagger = I$ $\longleftrightarrow$ **Consequence:** Reversible computation is natively mandatory.

2. SINGLE-QUBIT OPERATIONS & THE PAULI GROUP

Single-qubit gates manipulate state vectors on the complex Bloch Sphere, mapping transformations on the basis states $\{|0\rangle, |1\rangle\}$.

The Pauli Matrices & Hadamard Transformation:

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Superposition Generation: $H|0\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = |+\rangle$ and $H|1\rangle = \frac{|0\rangle-|1\rangle}{\sqrt{2}} = |-\rangle$

- **Phase Gates:** The S gate $\mathrm{diag}(1, i)$ executes a $\pi/2$ rotation, while the T gate $\mathrm{diag}(1, e^{i\pi/4})$ executes a critical $\pi/4$ rotation around the Z-axis.

3. MULTI-QUBIT ENTANGLING GATES

Multi-qubit systems scale via tensor products ($\otimes$). Controlled gates apply a target transformation conditioned strictly on the state of control qubits.

Controlled-NOT (CNOT) Operator: Flips the target qubit if and only if the control qubit is $|1\rangle$.

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{Mapping: } \begin{matrix} |00\rangle \to |00\rangle \\ |01\rangle \to |01\rangle \end{matrix} \text{ and } \begin{matrix} |10\rangle \to |11\rangle \\ |11\rangle \to |10\rangle \end{matrix}$$

Controlled-Phase (CZ) Operator: Injects a negative phase factor (-1) if and only if both system qubits occupy the $|11\rangle$ state. $\mathrm{CZ} = \mathrm{diag}(1, 1, 1, -1)$.

GENERATING THE 3-QUBIT GHZ STATE

Global coherence is built sequentially using a mix of single-qubit and multi-qubit entangling operations:

$$\text{Circuit Operations: } (\mathrm{CNOT}_{2\to3})(\mathrm{CNOT}_{1\to2})(H \otimes I \otimes I)|000\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}}$$

4. COMPUTATIONAL UNIVERSALITY

A finite set of quantum gates is classified as **Universal** if any arbitrary unitary operation can be approximated to any desired degree of accuracy ε.

1. **The Clifford Set:** Consisting of $\{H, S, \mathrm{CNOT}\}$, this set can efficiently be simulated classically (Gottesman-Knill Theorem).
2. **Clifford + T Set:** Adding the non-Clifford T gate breaks classical simulatability, yielding absolute universal quantum computational power.