WAVE FUNCTION EVANESCENCE & POTENTIAL BARRIER PENETRATION

QUANTUM TUNNELING

Boundary Boundary Matching, Transmission Coefficients, and Solid-State Systems

An axiomatic mathematical guide mapping the explicit wave mechanics that allow particles to bypass classical energy barriers.

1. THE PARADIGM SHIFT: DEFYING ABSOLUTE KINETIC BOUNDARIES

In classical mechanics, a particle of mass m and total energy E encountering a localized spatial potential energy barrier $V(x) = V_0$ is governed by strict energy conservation constraints. If $E < V_0$, the particle's classical kinetic energy becomes negative, forcing a deterministic reflection at the boundary interface. Quantum mechanics rejects this absolute barrier constraint by re-mapping particle dynamics into a continuous, non-local probability wave distribution.

2. EXPLICIT ANALYTICAL DERIVATION: THE FINITE RECTANGULAR BARRIER

Consider a one-dimensional potential profile where $V(x) = V_0$ for $0 \le x \le a$, and $V(x) = 0$ everywhere else. For a particle stream incident from the left in the energy regime where $E < V_0$, the time-independent Schrödinger equation partitions phase space into three distinct analytical regions:

Piecewise Wave Function Solutions:

$$\textbf{Region I } (x < 0): \quad \psi_{\mathrm{I}}(x) = Ae^{ikx} + Be^{-ikx} \qquad \text{where } k = \frac{\sqrt{2mE}}{\hbar}$$

$$\textbf{Region II } (0 \le x \le a): \quad \psi_{\mathrm{II}}(x) = Ce^{-\kappa x} + De^{\kappa x} \qquad \text{where } \kappa = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$\textbf{Region III } (x > a): \quad \psi_{\mathrm{III}}(x) = Fe^{ikx} \qquad \text{where } F \text{ is the transmission amplitude}$$

The Boundary Continuity Conditions: Continuity of the global wave function and its spatial derivative (ψ and $d\psi/dx$) at the matching interfaces $x = 0$ and $x = a$ locks the system coefficient matrix.

3. THE TRANSMISSION COEFFICIENT & THE EVANESCENT LIMIT

Evaluating the probability current density vectors across the boundaries determines the exact **Transmission Coefficient** ($T = |F/A|^2$), which quantifies the absolute probability of a single incident particle tunneling completely through the barrier:

$$T = \left[1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2(\kappa a)\right]^{-1}$$

When encountering thick or high barriers where $\kappa a \gg 1$, the hyperbolic sine function approaches an exponential profile ($\sinh(\kappa a) \approx \frac{1}{2}e^{\kappa a}$). This simplifies the transmission scaling to the fundamental **Evanescent Tunneling Law**:

$$T \approx \frac{16E(V_0 - E)}{V_0^2} \exp(-2\kappa a) \quad \Longrightarrow \quad \mathbf{T \propto \exp\left(-\frac{2a}{\hbar}\sqrt{2m(V_0 - E)}\right)}$$

The probability drops exponentially with barrier width (a) and the square root of the energy deficit ($V_0 - E$).

4. FUNDAMENTAL PHYSICAL APPLICATIONS & NATURAL PHENOMENA

Quantum tunneling is not an isolated subatomic oddity; it dictates major cosmological, chemical, and industrial hardware operations:

STELLAR FUSION AND ALPHA DECAY PHENOMENA

The core temperature of the Sun is classically too cold ($\sim 1.5 \times 10^7$ K) to permit hydrogen nuclei to overcome their mutual electrostatic Coulomb repulsion barrier.

George Gamow's Tunneling Factor: $\mathcal{P} \propto \exp\left(-\sqrt{\frac{E_G}{E}}\right)$

Stellar fusion relies entirely on continuous quantum tunneling through this Coulomb wall. The same mathematical framework reversed explains alpha particle decay lifetimes in heavy unstable radioactive isotopes.

1. **Scanning Tunneling Microscopy (STM):** Maps surface atomic structures with picometer resolution by sweeping a sharp conductive tip past a sample. Because the tunneling current tracks exponentially with the air gap ($I \propto T \propto e^{-2\kappa a}$), sub-angstrom distance changes trigger massive, clean electrical current shifts.
2. **Flash Memory Architectures:** Solid-state drives write and erase data bits using Fowler-Nordheim tunneling mechanics. Applying a localized electric field tilts the oxide potential barrier into a triangular shape, allowing electrons to slip into a floating gate to store charge data.