LIGHT-MATTER INTERACTION & QUANTUM CONTROL

RABI OSCILLATIONS & THE JC MODEL

Coherent Coaxial Dynamics of Atom-Cavity Coupled Systems

An axiomatic guide detailing semi-classical drives, fully quantized fields, and the Rotating Wave Approximation.

1. SEMI-CLASSICAL CONTROL: RABI OSCILLATIONS

When a two-level quantum system (qubit) interacts with a classical monochromatic electromagnetic drive, the state vector undergoes coherent periodic population transfers known as **Rabi Oscillations**.

The Driven Qubit Hamiltonian (Rotating Frame): By applying the Rotating Wave Approximation (RWA) to drop counter-rotating terms oscillating rapidly at 2ω, the system energy matrix under a drive frequency ω_d simplifies to:

$$H_{\text{rot}} = \frac{\hbar}{2}\Delta Z + \frac{\hbar}{2}\Omega X = \frac{\hbar}{2}\begin{pmatrix} \Delta & \Omega \\ \Omega & -\Delta \end{pmatrix}$$

Where: $\Delta = \omega_q - \omega_d$ is the drive detuning, Ω is the Rabi frequency (drive amplitude), and X, Z are Pauli matrices.

Transition Probability Evolution: Preparing the register in $|0\rangle$ at $t = 0$ and applying the drive yields a time-dependent probability of flipping the state to $|1\rangle$:

$$P_{0\to 1}(t) = \frac{\Omega^2}{\Omega_{\text{eff}}^2}\sin^2\left(\frac{\Omega_{\text{eff}} t}{2}\right) \qquad \text{where the effective frequency is } \Omega_{\text{eff}} = \sqrt{\Omega^2 + \Delta^2}$$

- **On-Resonance ($\Delta = 0$):** Ensures a complete population swap ($P_{\max} = 1$) at a pulse time of $t = \pi/\Omega$ (a π-pulse).
- **Off-Resonance ($\Delta \neq 0$):** Destructive interference restricts full population transfer ($P_{\max} < 1$), speeding up the oscillation velocity but minimizing maximum state-flip yield.

2. FULLY QUANTIZED INTERACTION: THE JAYNES-CUMMINGS HAMILTONIAN

The **Jaynes-Cummings (JC) Model** upgrades the classical drive to a fully quantized electromagnetic mode inside an optical or superconducting cavity, forming the mathematical backbone of Cavity Quantum Electrodynamics (CQED).

The Total Hamiltonian: Combining the bare qubit energy, the quantized photon field, and the dipole interaction yields:

$$H_{\text{JC}} = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right) + \frac{1}{2}\hbar\omega_q Z + \hbar g\left(a^\dagger\sigma_- + a\sigma_+\right)$$

Where $a^\dagger/a$ are the photon creation/annihilation operators, σ_+/σ_- are the qubit raising/lowering operators, and g is the vacuum coupling strength. The interaction terms conserve total excitation number $N = a^\dagger a + \frac{1}{2}Z$.

3. ENERGY SPLITTING: DRESSED STATE SUBSPACES

Because total excitation is conserved, the JC matrix diagonalizes into uncoupled two-dimensional manifolds spanned by the unperturbed basis blocks $\{|e, n\rangle, |g, n+1\rangle\}$.

THE DRESSED EIGENSTATES & VACUUM SPLITTING

Diagonalizing the n-photon interaction manifold yields the strongly coupled "dressed" eigenstates:

$$|+, n\rangle = \cos\theta_n\,|e, n\rangle + \sin\theta_n\,|g, n+1\rangle \qquad |-, n\rangle = -\sin\theta_n\,|e, n\rangle + \cos\theta_n\,|g, n+1\rangle$$

$$\text{Where the mixing angle matches: } \tan(2\theta_n) = \frac{2g\sqrt{n+1}}{\Delta}$$

The Vacuum Rabi Splitting: On resonance ($\Delta = 0$), the energy separation between these states expands to $\Delta E = 2\hbar g\sqrt{n+1}$. Crucially, even with an empty cavity ($n = 0$), a splitting of $2\hbar g$ persists due to interaction with vacuum fluctuations.

4. THE DISPERSIVE REGIME & QUBIT READOUT

When the qubit and cavity are far detuned relative to their coupling matrix ($|\Delta| = |\omega_q - \omega_r| \gg g$), actual real photon exchanges are suppressed. Applying a second-order Schrieffer-Wolff transformation reveals the dispersive Hamiltonian:

$$H_{\text{disp}} \approx \hbar\left(\omega_r + \chi Z\right) a^\dagger a + \frac{1}{2}\hbar\omega_q' Z \qquad \text{where the dispersive shift is } \chi = \frac{g^2}{\Delta}$$

The Computational Realization: The cavity resonance frequency shifts by $+\chi$ or $-\chi$ conditional on the internal state of the qubit ($Z = \pm 1$). Measuring this microwave transmission shift enables non-destructive dispersive qubit state readout.