BIPARTITE TENSOR SPACES & SINGULAR VALUE TRANSFORMATIONS

SCHMIDT DECOMPOSITION

Diagonalization of Composite Pure States and Entanglement Metrics

An axiomatic mathematical guide detailing coordinate transformations, Schmidt ranks, and reduced matrix operators.

1. THE FOUNDATIONAL THEOREM: BIPARTITE STATE DIAGONALIZATION

The **Schmidt Decomposition Theorem** states that any arbitrary pure state $|\psi\rangle_{AB}$ belonging to a composite bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written in a strictly diagonalized form across specially chosen local orthonormal bases.

The Schmidt Representation Formula: Let d_A and d_B be the dimensions of systems A and B. There exist orthonormal bases $\{|i_A\rangle\}$ for $\mathcal{H}_A$ and $\{|i_B\rangle\}$ for $\mathcal{H}_B$ such that:

$$|\psi\rangle_{AB} = \sum_{i=1}^{k} \lambda_i |i_A\rangle |i_B\rangle \qquad \text{where } \lambda_i > 0 \quad \text{and} \quad \sum_{i=1}^{k} \lambda_i^2 = 1$$

Where: $k \leq \min(d_A, d_B)$ is an invariant integer called the **Schmidt Rank** (or Schmidt number), and λ_i are the positive scalar **Schmidt Coefficients**.

2. THE ALGEBRAIC ENGINE: SINGULARITY MAPPING VIA SVD

The Schmidt decomposition is the direct translation of classical **Singular Value Decomposition (SVD)** into the notation of quantum mechanics.

The Derivation Pipeline: Expand an arbitrary state vector across any standard background basis: $|\psi\rangle_{AB} = \sum_{j,l} C_{jl} |j_A\rangle |l_B\rangle$, where C is a complex $d_A \times d_B$ coefficient matrix.

- **Matrix Factorization:** Compute the SVD of the coefficient matrix: $C = U\Sigma V^\dagger \implies C_{jl} = \sum_i U_{ji}\Sigma_{ii}V_{li}^*$.
- **Basis Rotation:** Define the new local target bases by applying the unitary matrices to the original vectors: $|i_A\rangle = \sum_j U_{ji} |j_A\rangle$ and $|i_B\rangle = \sum_l V_{li}^* |l_B\rangle$.
- **The Identity Convergence:** The singular values Σ_{ii} match the positive real Schmidt coefficients λ_i exactly.

3. SUBSYSTEM MANIFESTATION: EIGENVALUES OF REDUCED OPERATORS

The Schmidt parameters dictate the exact spectrum of local systems when evaluated under a partial trace operator Tr_B or Tr_A.

SUBSYSTEM OPERATOR CORRELATION

Tracing out system B to isolate the reduced density matrix ρ_A collapses all off-diagonal cross terms, yielding a clean spectral layout:

$$\rho_A = \mathrm{Tr}_B(|\psi\rangle_{AB}\langle\psi|) = \sum_i \lambda_i^2 |i_A\rangle\langle i_A| \qquad \rho_B = \mathrm{Tr}_A(|\psi\rangle_{AB}\langle\psi|) = \sum_i \lambda_i^2 |i_B\rangle\langle i_B|$$

The Core Consequence: The squares of the Schmidt coefficients (λ_i^2) form the shared, identical eigenvalue spectrum for both reduced density matrices, proving that subsystem mixedness metrics are perfectly balanced.

4. QUANTIFYING RESOURCE ENTANGLEMENT BOUNDS

The structural geometry of the Schmidt decomposition makes it the definitive test for classifying bipartite pure-state entanglement:

1. **Separability Condition ($k = 1$):** If the Schmidt rank equals exactly 1, the vector scales as a simple product state $|\psi\rangle_{AB} = |1_A\rangle |1_B\rangle$. The state contains zero non-local correlations.
2. **Entangled Condition ($k > 1$):** If multiple coefficients exist, the state cannot be factored classically. The shared spectrum calculates the absolute **Entropy of Entanglement** (E) via the Von Neumann entropy of ρ_A:

$$E(|\psi\rangle_{AB}) = -\sum_{i=1}^{k} \lambda_i^2 \log_2(\lambda_i^2) \quad \implies \quad \text{Maximally Mixed Bound: } \mathbf{E_{max} = \log_2(\min(d_A, d_B))}$$

5. THE MULTI-PARTITE GOTCHA: THE LIMIT OF TWO PARTIES

A major conceptual pitfall is assuming that the Schmidt decomposition generalizes naturally to higher multi-partite configurations ($N \geq 3$ qubits). It does not.

$$|\mathrm{W}\rangle_3 = \frac{1}{\sqrt{3}}(|100\rangle + |012\rangle + |001\rangle) \quad \longleftrightarrow \quad \textbf{Cannot be diagonalized synchronously!}$$

For three or more systems, it is mathematically impossible to construct localized transformations that simplify the global state vector into a single index summation map $\sum_i \lambda_i |i_A\rangle |i_B\rangle |i_C\rangle$. Multi-partite spaces split into distinct SLOCC classes (like GHZ vs. W), escaping single-diagonal matrix operations.