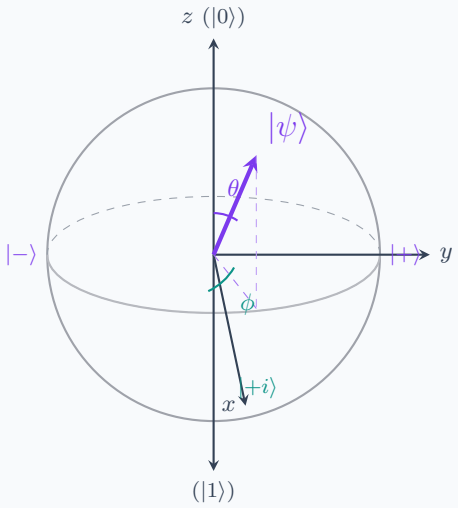
GEOMETRIC MAPPING OF TWO-LEVEL QUANTUM STATE SPACES

THE BLOCH SPHERE

Spherical Coordinates, Pure vs. Mixed States, and Universal Rotations

An axiomatic mathematical flyer mapping single-qubit kinematics directly onto a 3D unit sphere geometric layout.



1. MATHEMATICAL MAPPING: THE STATE PARAMETRIZATION

Because the total probability of a qubit must be perfectly normalized ($\alpha^2 + \beta^2 = 1$), and an overall global phase is physically unmeasurable, any arbitrary pure state vector $|\psi\rangle$ inside the Hilbert space $\mathcal{H}_2$ can be uniquely defined using two geometric angles on a unit sphere.

The Spherical Coordinate Wave Function Formula:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle$$

Where: $\theta \in [0, \pi]$ is the colatitude angle matching the z-axis, and $\phi \in [0, 2\pi]$ is the azimuthal phase angle matching the x-y plane.

2. THE DENSITY MATRIX ALIGNMENT: THE BLOCH VECTOR ($\vec{R}$)

When scaling calculations to open systems containing environmental decoherence, state metrics switch from 2D vector coordinates to a 3D operational vector $\vec{r} = (x, y, z)^T$ inside the density matrix ρ:

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right) = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix} \qquad \text{where } \begin{aligned} x &= \mathrm{Tr}(\rho X) = \sin\theta\cos\phi \\ y &= \mathrm{Tr}(\rho Y) = \sin\theta\sin\phi \\ z &= \mathrm{Tr}(\rho Z) = \cos\theta \end{aligned}$$

- **The Sphere Surface ($\|\vec{r}\| = 1$):** Every point on the physical shell maps an absolute **pure state** ($\mathrm{Tr}(\rho^2) = 1$).
- **The Sphere Interior ($\|\vec{r}\| < 1$):** Inner nodes map degraded **mixed states** that have experienced dephasing.
- **The Center Origin ($\|\vec{r}\| = 0$):** Represents the **maximally mixed state** $\rho = \frac{1}{2}I$, possessing zero quantum phase memory.

3. CIRCUIT GATE MECHANICS AS RIGID 3D ROTATIONS

The Bloch sphere turns abstract single-qubit quantum gate matrices into intuitive rigid geometric rotations of the state vector around chosen axes by specific angles.

THE UNIVERSAL ROTATION ENGINE

Any unitary transformation can be parameterized as an exponential rotation matrix matching a structural axis vector $\hat{n}$:

$$R_{\hat{n}}(\alpha) = \exp\left(-i\frac{\alpha}{2}\hat{n}\cdot\vec{\sigma}\right) = \cos\left(\frac{\alpha}{2}\right)I - i\sin\left(\frac{\alpha}{2}\right)(\hat{n}\cdot\vec{\sigma})$$

The Hardware Realization: The Pauli gate $X = R_{\hat{x}}(\pi)$ flips the vector 180° around the x-axis (NOT operation). The Hadamard gate $H = R_{\hat{x}+\hat{z}}(\pi)$ takes the north pole ($|0\rangle$) and swings it down to the equator ($|+\rangle$), converting basis attributes into global phase superpositions.

4. THE HARD LIMIT: THE MULTI-QUBIT EXPLOSION

A massive conceptual trap is assuming that larger registers can be modeled by packing extra vectors onto the Bloch sphere. They cannot. The Bloch sphere is **strictly limited to a single qubit ($N = 1$)**.

$$\mathcal{H}_{\text{composite}} = \mathcal{H}_2^{\otimes N} \implies \text{Dimension scales as } 2^N \quad \longleftrightarrow \quad \textbf{Escapes 3D physical boundaries!}$$

As soon as two qubits become entangled (such as a Bell pair or a GHZ state), their non-local correlations exist within the joint tensor product space, which cannot be broken down or visualised using independent single-particle vectors.