CLASSICAL ESTIMATION THEORY & QUANTUM METROLOGY

THE CRÁMER-RAO BOUND

Fisher Information Matrices and Fundamental Parameter Estimation Limits

An axiomatic guide detailing classical estimation boundaries, Quantum Fisher Information, and the Heisenberg Limit.

1. THE CLASSICAL BOUNDARY: FUNDAMENTAL ESTIMATION LIMITS

The **Crámer-Rao Bound (CRB)** establishes the absolute lower bound on the variance of any unbiased estimator $\hat{\theta}$ attempting to extract an unknown deterministic parameter θ from a statistical distribution of experimental measurement outcomes x.

The Classical Crámer-Rao Inequality: For an unbiased estimator evaluating N independent measurements, the variance satisfies:

$$\mathrm{Var}(\hat{\theta}) \geq \frac{1}{F(\theta)} \qquad \text{where } F(\theta) \text{ is the } \textbf{Classical Fisher Information (CFI)}$$

Mathematical CFI Formulations: Derived from the likelihood function $P(x|\theta)$ of the dataset:

$$F(\theta) = \mathbb{E}\left[\left(\frac{\partial \ln P(x|\theta)}{\partial \theta}\right)^2\right] = -\int P(x|\theta)\frac{\partial^2 \ln P(x|\theta)}{\partial \theta^2} dx$$

Physical Scaling Properties:

- **The Standard Quantum Limit (SQL):** For N independent, identically distributed measurement samples, the Fisher information tracks linearly ($F_{\text{total}} = NF_0$). This forces the statistical precision to scale as: $\Delta\theta \geq \frac{1}{\sqrt{NF_0}}$, defining the classical sampling boundary.
- **Efficiency:** An estimator that directly matches this mathematical lower limit is classified as **efficient** (e.g., Maximum Likelihood Estimators asymptotically achieve efficiency).

2. THE QUANTUM EXTENSION: QUANTUM FISHER INFORMATION

In quantum mechanics, parameters like phase shifts (ϕ) are not direct Hermitian observables; they are encoded into state operators via unitary evolution $\rho(\phi) = e^{-iH\phi}\rho_0 e^{iH\phi}$. The **Quantum Crámer-Rao Bound (QCRB)** maximizes precision across **all possible** quantum measurement strategies.

The Symmetric Logarithmic Derivative (SLD): To convert quantum operators into statistical differentials, define the Hermitian operator L_ϕ implicitly via the Lyapunov matrix equation:

$$\frac{\partial \rho(\phi)}{\partial \phi} = \frac{\rho(\phi)L_\phi + L_\phi \rho(\phi)}{2} \qquad \Longrightarrow \qquad \textbf{Quantum Fisher Information (QFI):} \quad F_Q = \mathrm{Tr}(\rho(\phi)L_\phi^2)$$

THE QUANTUM ESTIMATION BOUNDARY

The ultimate lower bound on parameter variance optimized across all measurement operators and state preparations satisfies:

$$\mathrm{Var}(\hat{\phi}) \geq \frac{1}{F_Q(\phi)} \qquad \leq \qquad \frac{1}{F(\phi)}$$

Pure State Simplification: For a pure state $\rho = |\psi\rangle\langle\psi|$ evolving under Hamiltonian H, the QFI matches four times the structural variance of the generator: $\mathbf{F_Q = 4\,\Delta H^2 = 4(\langle\psi|\,H^2\,|\psi\rangle - \langle\psi|\,H\,|\psi\rangle^2)}$.

3. BREAKING CLASSICAL LIMITS: THE HEISENBERG SCALING

The primary objective of quantum metrology is leveraging non-classical correlations (global quantum coherence) to maximize F_Q, bypassing the classical SQL bottleneck.

1. **Independent Product States:** Preparing N uncorrelated qubits yields a maximum QFI scaling linearly: $F_Q = N$. This traps measurement precision strictly at the standard classical limit: $\Delta\phi \geq 1/\sqrt{N}$.
2. **Maximally Entangled GHZ States:** Evaluating the N-qubit state $|\mathrm{GHZ}\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes N} + |1\rangle^{\otimes N})$ under the collective Hamiltonian $H = \sum_{i=1}^{N} Z_i/2$ yields a highly amplified variance calculation:

$$\Delta H^2 = \langle \mathrm{GHZ} | H^2 | \mathrm{GHZ} \rangle - \langle \mathrm{GHZ} | H | \mathrm{GHZ} \rangle^2 = \frac{N^2}{4} - 0 \implies \mathbf{F_Q = N^2}$$

Plugging this result into the QCRB reveals the absolute physical limit allowed by quantum mechanics—the **Heisenberg Limit**:

$\Delta\phi_{\mathrm{HL}} \geq \frac{1}{N}$ (An exponential quadratic improvement over classical configurations)